Modular curves, C*-algebras, and chaotic cosmology

Matilde Marcolli

Max–Planck Institut für Mathematik, Bonn, Germany;
marcolli@mpim-bonn.mpg.de

We make some brief remarks on the relation of the mixmaster universe model of chaotic cosmology to the geometry of modular curves and to noncommutative geometry. We show that the full dynamics of the mixmaster universe is equivalent to the geodesic flow on the modular curve $X_{\Gamma_0(2)}$. We then consider a special class of solutions, with bounded number of cycles in each Kasner era, and describe their dynamical properties (invariant density, Lyapunov exponent, topological pressure). We relate these properties to the noncommutative geometry of a moduli space of such solutions, which is given by a Cuntz–Krieger C*-algebra.

1 Modular curves

Let G be a finite index subgroup of $\Gamma = \mathrm{PGL}(2,\mathbb{Z})$, and let X_G denote the quotient $X_G = G\backslash\mathbb{H}^2$, where $\mathbb{H}^2$ is the 2-dimensional real hyperbolic plane, namely the upper half plane $\{z \in \mathbb{C} : \Im z > 0\}$ with the metric $ds^2 = |dz|^2/(\Im z)^2$. Equivalently, we identify $\mathbb{H}^2$ with the Poincaré disk $\{z : |z| < 1\}$ with the metric $ds^2 = 4|dz|^2/(1-|z|^2)^2$.

Let $\mathbb{P}$ denote the coset space $\mathbb{P} = \Gamma/G$. We can write the quotient X_G equivalently as $X_G = \Gamma\backslash(\mathbb{H}^2 \times \mathbb{P})$. The quotient space X_G has the structure of a non-compact Riemann surface, which can be compactified by adding cusp points at infinity:

$$\bar{X}_G = G\backslash(\mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q})) \simeq \Gamma\backslash\left((\mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q})) \times \mathbb{P}\right). \tag{1}$$

In particular, we consider the congruence subgroups $G = \Gamma_0(p)$, with p a prime, given by matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying $c \equiv 0 \mod p$. In fact, for our purposes, we are especially interested in the case $p = 2$.

1.1 Shift operator and dynamics

If we consider the boundary $\mathbb{P}^1(\mathbb{R})$ of $\mathbb{H}^2$, the arguments given in [9] and [10] show that the quotient $\Gamma\backslash(\mathbb{P}^1(\mathbb{R})\times\mathbb{P})$, better interpreted as a "noncommutative space", gives rise to a compactification of the modular curve X_G with a structure richer than the ordinary algebro-geometric compactification by cusp points.

In [9] this was described in terms of the following dynamical system, generalizing the classical Gauss shift of the continued fraction expansion:

$$T : [0,1]\times\mathbb{P} \to [0,1]\times\mathbb{P}$$

$$T(x,t) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} \cdot t \right). \tag{2}$$

In fact, the quotient space of $\mathbb{P}^1(\mathbb{R})\times\mathbb{P}$ by the $\mathrm{PGL}(2,\mathbb{Z})$ can be identified with the space of orbits of the dynamical system T on $[0,1]\times\mathbb{P}$.

We use the notation $x=[k_1,k_2,\ldots,k_n,\ldots]$ for the continued fraction expansion of the point $x\in[0,1]$, and we denote by $p_n(x)/q_n(x)$ the convergents of the continued fraction, with $p_n(x)$ and $q_n(x)$ the successive numerators and denominators. It is then easy to verify that the shift T acts on x by shifting the continued fraction expansion, $T[k_1,k_2,\ldots,k_n,\ldots]=[k_2,k_3,\ldots,k_{n+1},\ldots]$. The element $g_n(x)^{-1}$, with

$$g_n(x) = \begin{pmatrix} p_{k-1}(x) & p_k(x) \\ q_{k-1}(x) & q_k(x) \end{pmatrix} \in \Gamma,$$

acts on $[0,1]\times\mathbb{P}$ as T^n.

The Lyapunov exponent

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log|(T^n)'(x)|$$
$$= 2\lim_{n\to\infty} \frac{1}{n} \log q_n(x)$$

measures the exponential rate of divergence of nearby orbits, hence it provides a measure of how chaotic the dynamics is.

For the shift of the continued fraction expansion, the results of [14] show that $\lambda(x)=\pi^2/(6\log 2)=\lambda_0$ almost everywhere with respect to the Lebesgue measure on $[0,1]$ and counting measure on $\mathbb{P}$. On the other hand, the Lyapunov exponent takes all values $\lambda(x)\in[\lambda_0,\infty)$. The unit interval correspondingly splits as a union of T-invariant level sets of λ (Lyapunov spectrum) of varying Hausdorff dimension, plus an exceptional set where the limit defining λ does not converge.

2 Mixmaster universe

An important problem in cosmology is understanding how anisotropy in the early universe affects the long time evolution of space-time. This problem is relevant to the study of the beginning of galaxy formation and in relating the anisotropy of the background radiation to the appearance of the universe today.

We follow [2] (*cf.* also [13]) for a brief summary of anisotropic and chaotic cosmology. The simplest significant cosmological model that presents strong anisotropic properties is given by the Kasner metric

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{3}$$

where the exponents p_i are constants satisfying $\sum p_i = 1 = \sum_i p_i^2$. Notice that, for $p_i = d\log g_{ii}/d\log g$, the first constraint $\sum_i p_i = 1$ is just the condition that $\log g_{ij} = 2\alpha\delta_{ij} + \beta_{ij}$ for a traceless β, while the second constraint $\sum_i p_i^2 = 1$ amounts to the condition that, in the Einstein equations written in terms of α and β_{ij},

$$\left(\frac{d\alpha}{dt}\right)^2 = \frac{8\pi}{3}\left(T^{00} + \frac{1}{16\pi}\left(\frac{d\beta_{ij}}{dt}\right)^2\right)$$

$$e^{-3\alpha}\frac{d}{dt}\left(e^{3\alpha}\frac{d\beta_{ij}}{dt}\right) = 8\pi\left(T_{ij} - \frac{1}{3}\delta_{ij}T_{kk}\right),$$

the term T^{00} is negligible with respect to the term $(d\beta_{ij}/dt)^2/16\pi$, which is the "effective energy density" of the anisotropic motion of empty space, contributing together with a matter term to the Hubble constant.

Around 1970, Belinsky, Khalatnikov, and Lifshitz introduced a cosmological model (*mixmaster universe*) where they allowed the exponents p_i of the Kasner metric to depend on a parameter u,

$$\begin{aligned} p_1 &= \tfrac{-u}{1+u+u^2} \\ p_2 &= \tfrac{1+u}{1+u+u^2} \\ p_3 &= \tfrac{u(1+u)}{1+u+u^2} \end{aligned} \tag{4}$$

Since for fixed u the model is given by a Kasner space-time, the behavior of this universe can be approximated for certain large intervals of time by a Kasner metric. In fact, the evolution is divided into Kasner eras and each era into cycles. During each era the mixmaster universe goes through a volume compression. Instead of resulting in a collapse, as with the Kasner metric, high negative curvature develops resulting in a bounce (transition to a new era) which starts again a behavior approximated by a Kasner metric, but with a different value of the parameter u. Within each era, most of the volume compression is due to the scale factors along one of the space axes, while the

other scale factors alternate between phases of contraction and expansion. These alternating phases separate cycles within each era.

More precisely, we are considering a metric

$$ds^2 = -dt^2 + a(t)dx^2 + b(t)dy^2 + c(t)dz^2, \tag{5}$$

generalizing the Kasner metric (3). We still require that (5) admits $SO(3)$ symmetry on the space-like hypersurfaces, and that it presents a singularity at $t \to 0$. In terms of logarithmic time $d\Omega = -\frac{dt}{abc}$, the *mixmaster universe* model of Belinsky, Khalatnikov, and Lifshitz admits a discretization with the following properties:

1. The time evolution is divided in Kasner eras $[\Omega_n, \Omega_{n+1}]$, for $n \in \mathbb{Z}$. At the beginning of each era we have a corresponding discrete value of the parameter $u_n > 1$ in (4).
2. Each era, where the parameter u decreases with growing Ω, can be subdivided in cycles corresponding to the discrete steps u_n, $u_n - 1$, $u_n - 2$, etc. A change $u \to u - 1$ corresponds, after acting with the permutation (12)(3) on the space coordinates, to changing u to $-u$, hence replacing contraction with expansion and conversely. Within each cycle the space-time metric is approximated by the Kasner metric (3) with the exponents p_i in (4) with a fixed value of $u = u_n - k > 1$.
3. An era ends when, after a number of cycles, the parameter u_n falls in the range $0 < u_n < 1$. Then the bouncing is given by the transition $u \to 1/u$ which starts a new series of cycles with new Kasner parameters and a permutation (1)(23) of the space axis, in order to have again $p_1 < p_2 < p_3$ as in (4).

Thus, the transition formula relating the values u_n and u_{n+1} of two successive Kasner eras is

$$u_{n+1} = \frac{1}{u_n - [u_n]},$$

which is exactly the shift of the continued fraction expansion, $Tx = 1/x - [1/x]$, with $x_{n+1} = Tx_n$ and $u_n = 1/x_n$.

3 Geodesics and universes

The previous observation is the key to a geometric description of solutions of the mixmaster universe in terms of geodesics on a modular curve (Manin–Marcolli [9]):

Theorem 3.1 *Consider the modular curve $X_{\Gamma_0(2)}$. Each infinite geodesic γ on $X_{\Gamma_0(2)}$ not ending at cusps determines a mixmaster universe.*

Proof. An infinite geodesic on $X_{\Gamma_0(2)}$ is the image under the quotient map

$$\pi_\Gamma : \mathbb{H}^2 \times \mathbb{P} \to \Gamma\backslash(\mathbb{H}^2 \times \mathbb{P}) \cong X_G,$$

where $\Gamma = \mathrm{PGL}(2,\mathbb{Z})$, $G = \Gamma_0(2)$, and $\mathbb{P} = \Gamma/G \cong \mathbb{P}^1(\mathbb{F}_2) = \{0,1,\infty\}$, of an infinite geodesic on $\mathbb{H}^2 \times \mathbb{P}$ with ends on $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$. We consider the elements of $\mathbb{P}^1(\mathbb{F}_2)$ as labels assigned to the three space axes, according to the identification

$$\begin{array}{ll} 0 = [0:1] & \mapsto z \\ \infty = [1:0] & \mapsto y \\ 1 = [1:1] & \mapsto x. \end{array} \tag{6}$$

Any geodesic not ending at cups can be coded in terms of data (ω^-, ω^+, s), where $(\omega^\pm, s)$ are the endpoints in $\mathbb{P}^1(\mathbb{R}) \times \{s\}$, $s \in \mathbb{P}$, with $\omega^- \in (-\infty, -1]$ and $\omega^+ \in [0,1]$. In terms of the continued fraction expansion, we can write

$$\begin{array}{l} \omega^+ = [k_0, k_1, \ldots k_r, k_{r+1}, \ldots] \\ \omega^- = [k_{-1}; k_{-2}, \ldots, k_{-n}, k_{-n-1}, \ldots]. \end{array}$$

The shift acts on these data by

$$T(\omega^+, s) = \left(\frac{1}{\omega^+} - \left[\frac{1}{\omega^+}\right], \begin{pmatrix} -[1/\omega^+] & 1 \\ 1 & 0 \end{pmatrix} \cdot s \right)$$

$$T(\omega^-, s) = \left(\frac{1}{\omega^- + [1/\omega^+]}, \begin{pmatrix} -[1/\omega^+] & 1 \\ 1 & 0 \end{pmatrix} \cdot s \right).$$

Geodesics on $X_{\Gamma_0(2)}$ can be identified with the orbits of T on the set of data (ω, s).

The data (ω, s) determine a mixmaster universe, with the $k_n = [u_n] = [1/x_n]$ in the Kasner eras, and with the transition between subsequent Kasner eras given by $x_{n+1} = Tx_n \in [0,1]$ and by the permutation of axes induced by the transformation

$$\begin{pmatrix} -k_n & 1 \\ 1 & 0 \end{pmatrix}$$

acting on $\mathbb{P}^1(\mathbb{F}_2)$. It is easy to verify that, in fact, this acts as the permutation $0 \mapsto \infty$, $1 \mapsto 1$, $\infty \mapsto 0$, if k_n is even, and $0 \mapsto \infty$, $1 \mapsto 0$, $\infty \mapsto 1$ if k_n is odd, that is, after the identification (6), as the permutation $(1)(23)$ of the space axes (x, y, z), if k_n is even, or as the product of the permutations $(12)(3)$ and $(1)(23)$ if k_n is odd. This is precisely what is obtained in the mixmaster universe model by the repeated series of cycles within a Kasner era followed by the transition to the next era.

Data (ω, s) and $T^m(\omega, s)$, $m \in \mathbb{Z}$, determine the same solution up to a different choice of the initial time.

There is an additional time-symmetry in this model of the evolution of mixmaster universes (*cf.* [2]). In fact, there is an additional parameter δ_n in the system, which measures the initial amplitude of each cycle. It is shown in [2] that this is governed by the evolution of a parameter

$$v_n = \frac{\delta_{n+1}(1 + u_n)}{1 - \delta_{n+1}}$$

which is subject to the transformation across cycles $v_{n+1} = [u_n] + v_n^{-1}$. By setting $y_n = v_n^{-1}$ we obtain

$$y_{n+1} = \frac{1}{(y_n + [1/x_n])},$$

hence we see that we can interpret the evolution determined by the data (ω^+, ω^-, s) with the shift T either as giving the complete evolution of the u-parameter towards and away from the cosmological singularity, or as giving the simultaneous evolution of the two parameters (u, v) while approaching the cosmological singularity.

This in turn determines the complete evolution of the parameters (u, δ, Ω), where Ω_n is the starting time of each cycle. For the explicit recursion $\Omega_{n+1} = \Omega_{n+1}(\Omega_n, x_n, y_n)$ see [2].

Notice that, unlike the description of the full mixmaster dynamics given, for instance, in [2], we *include* the alternation of the space axes at the end of cycles and eras as part of the dynamics, which is precisely what determines the choice of the congruence subgroup. This also introduces a slight modification of some of the invariants associated to the mixmaster dynamics. For instance, it is proved in [9] that there is a unique T–invariant measure on $[0,1] \times \mathbb{P}$, which is given by the Gauss density on $[0,1]$ and the counting measure δ on $\mathbb{P}$:

$$d\mu(x,s) = \frac{\delta(s)\, dx}{3\log(2)\,(1+x)}, \tag{7}$$

which reduces to the Gauss density for the shift of the continued fraction on $[0,1]$ when integrated in the $\mathbb{P}$ direction. In particular, as observed in [9], the form (7) of the invariant measure implies that the alternation of the space axes is uniform over the time evolution, namely the three axes provide the scale factor responsible for volume compression with equal frequencies.

4 Controlled pulse universes

The interpretation of solutions in terms of geodesics provides a natural way to single out and study certain special classes of solutions on the basis of their geometric properties. Physically, such special classes of solutions exhibit different behaviors approaching the cosmological singularity.

For instance, the data (ω^+, s) corresponding to an eventually periodic sequence $k_0 k_1 \dots k_m \dots$ of some period $\overline{a_0 \dots a_\ell}$ correspond to those geodesics on $X_{\Gamma_0(2)}$ that asymptotically wind around the closed geodesic identified with the doubly infinite sequence $\dots a_0 \dots a_\ell a_0 \dots a_\ell \dots$. Physically, these universes exhibit a pattern of cycles that recurs periodically after a finite number of Kasner eras.

In the following we concentrate on another special class of solutions, which we call *controlled pulse universes*. These are the mixmaster universes for which there is a fixed upper bound N to the number of cycles in each Kasner era.

In terms of the continued fraction description, these solutions correspond to data (ω^+, s) with ω^+ in the Hensley Cantor set $E_N \subset [0,1]$. The set E_N is given by all points in $[0,1]$ with all digits in the continued fraction expansion bounded by N (*cf.* [6]). In more geometric terms, one considers geodesics on the modular curve $X_{\Gamma_0(2)}$ that wander only a finite distance into the cusps.

4.1 Dynamical properties

It is well known that a very effective technique for the study of dynamical properties such as topological pressure and invariant densities is given by transfer operator methods. These have already been applied successfully to the case of the mixmaster universe, *cf.* [11]. In our setting, for the full mixmaster dynamics that includes alternation of space axes, the Perron-Frobenius operator for the shift (2) is of the form

$$(\mathcal{L}_\beta f)(x,s) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^\beta} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot s \right).$$

This yields the density of the invariant measure (7) satisfying $\mathcal{L}_2 f = f$. The top eigenvalue η_β of $\mathcal{L}_\beta$ is related to the topological pressure by $\eta_\beta = \exp(P(\beta))$. This can be estimated numerically, using the technique of [1] and the integral kernel operator representation of §1.3 of [9].

We now restrict our attention to the case of controlled pulse universes. Since these form sets of measure zero in the measure (7), they support exceptional values of such dynamical invariants as Lyapunov exponent, topological pressure, entropy. In fact, for a fixed bound N on the number of cycles per era, we are considering the dynamical system (2)

$$T : E_N \times \mathbb{P} \to E_N \times \mathbb{P}.$$

For this map, the Perron-Frobenius operator is of the form

$$(\mathcal{L}_{\beta,N} f)(x,s) = \sum_{k=1}^{N} \frac{1}{(x+k)^\beta} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot s \right). \tag{8}$$

It is proved in [10] that this operator still has a unique invariant measure μ_N, whose density satisfies $\mathcal{L}_{2\dim_H(E_N),N} f = f$, with

$$\dim_H(E_N) = 1 - \frac{6}{\pi^2 N} - \frac{72 \log N}{\pi^4 N^2} + O(1/N^2)$$

the Hausdorff dimension of the Cantor set E_N. Moreover, the top eigenvalue η_β of $\mathcal{L}_{\beta,N}$ is related to the Lyapunov exponent by

$$\lambda(x) = 2 \frac{d}{d\beta} \eta_\beta |_{\beta = 2\dim_H(E_N)},$$

for μ_N-almost all $x \in E_N$, *cf.* [10].

5 Non-commutative spaces

A consequence of this characterization of the time evolution in terms of the dynamical system (2) is that we can study global properties of suitable *moduli spaces* of mixmaster universes. For instance, the moduli space for time evolutions of the u-parameter approaching the cosmological singularity as $\Omega \to \infty$ is given by the quotient of $[0,1] \times \mathbb{P}$ by the action of the shift T. Similarly, the moduli spaces that correspond to controlled pulse universes are the quotients of $E_N \times \mathbb{P}$ by the action of the shift T. It is easy to see that such quotients are not well behaved as a topological spaces, which makes it difficult to study their global properties in the context of classical geometry. However, non-commutative geometry in the sense of Connes [4] provides the correct framework for the study of such spaces. The occurrence in physics of non-commutative spaces as moduli spaces is not a new phenomenon. A well known example is the moduli space of Penrose tilings (*cf.* [4]), which plays an important role in the mathematical theory of quasi-crystals.

We consider here only the case of controlled pulse universes. In this case, the dynamical system T is a subshift of finite type which can be described by the Markov partition

$$\mathcal{A}_N = \{((k,t),(\ell,s)) | U_{k,t} \subset T(U_{\ell,s})\},$$

for $k, \ell \in \{1, \dots, N\}$, and $s, t \in \mathbb{P}$, with sets $U_{k,t} = U_k \times \{t\}$, where $U_k \subset E_N$ are the clopen subsets where the local inverses of T are defined,

$$U_k = \left[\frac{1}{k+1}, \frac{1}{k}\right] \cap E_N.$$

This Markov partition determines a matrix A_N, with entries $(A_N)_{kt,\ell s} = 1$ if $U_{k,t} \subset T(U_{\ell,s})$ and zero otherwise.

Lemma 5.1 *The* 3×3 *submatrices* $A_{k\ell} = (A_{(k,t),(\ell,s)})_{s,t \in \mathbb{P}}$ *of the matrix* A_N *are of the form*

$$A_{k\ell} = \begin{cases} M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \ell = 2m \\ \\ M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \ell = 2m+1 \end{cases}$$

The matrix A_N *is irreducible and aperiodic.*

Proof. The condition $U_{k,t} \subset T(U_{\ell,s})$ is equivalently written as the condition that

$$\begin{pmatrix} 0 & 1 \\ 1 & \ell \end{pmatrix} \cdot s = t.$$

We then proceed as in the proof of Theorem 3.1 and notice that the transformation

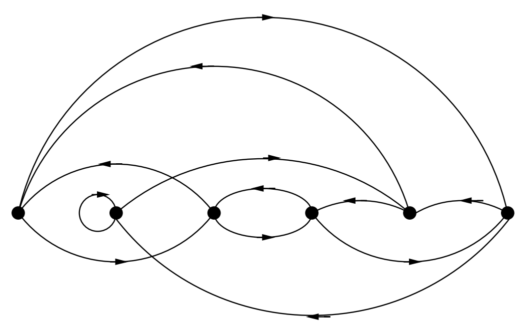
$$\begin{pmatrix} 0 & 1 \\ 1 & \ell \end{pmatrix}$$

acts on $\mathbb{P}^1(\mathbb{F}_2)$, under the identification (6), as the permutation $0 \mapsto \infty$, $1 \mapsto 1$, $\infty \mapsto 0$, when ℓ is even, and $\infty \mapsto 0$, $0 \mapsto 1$, $1 \mapsto \infty$ if ℓ is odd.

Irreducibility of A_N means that the corresponding directed graph is strongly connected, namely any two vertices are connected by an oriented path of edges. Since the matrix A_N has the form

$$A_N = \begin{pmatrix} M_1 & M_1 & M_1 & \cdots & M_1 \\ M_2 & M_2 & M_2 & \cdots & M_2 \\ M_1 & M_1 & M_1 & \cdots & M_1 \\ \vdots & & & \cdots & \vdots \end{pmatrix},$$

irreducibility follows from the irreducibility of A_2, which corresponds to the directed graph illustrated in the Figure. This graph also shows that the matrix



A_N is aperiodic. In fact, the period is defined as the gcd of the lengths of the closed directed paths and the matrix is called aperiodic when the period is equal to one.

As a non-commutative space associated to the Markov partition we consider the Cuntz–Krieger C*-algebra $\mathcal{O}_{A_N}$ (*cf.* [5]), which is the universal C*-algebra generated by partial isometries S_{kt} satisfying the relations

$$\sum_{(k,t)} S_{kt} S_{kt}^* = 1,$$

$$S_{\ell s}^* S_{\ell s} = \sum_{(k,t)} A_{(k,t),(\ell,s)} S_{kt} S_{kt}^*.$$

It is a well known fact that the structure of this C*-algebra reflect properties of the dynamics of the shift T. For example, one can recover the Bowen–Franks invariant of the dynamical system T from the K-theory of a Cuntz–Krieger C*-algebra $\mathcal{O}_A$ (*cf.* [5]). Moreover, in our set of examples, information on the dynamical properties of the shift T and the Perron–Frobenius operator (8) can be derived from the KMS states for the C^*-algebras $\mathcal{O}_{A_N}$ with respect to suitable time evolutions.

5.1 Time evolution and KMS states

Recall that a state φ on a unital C^*-algebra $\mathcal{A}$ is a continuous linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying $\varphi(a^* a) \geq 0$ and $\varphi(1) = 1$. Let σ_t be an action of $\mathbb{R}$ on $\mathcal{A}$ by automorphisms. A state φ satisfies the KMS condition at inverse temperature β if for any $a, b \in \mathcal{A}$ there exists a bounded holomorphic function $F_{a,b}$ continuous on $0 \leq Im(z) \leq \beta$, such that, for all $t \in \mathbb{R}$,

$$F_{a,b}(t) = \varphi(a\, \sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b\,)a) \tag{9}$$

Equivalently, the KMS condition (9) is expressed as the relation

$$\varphi(\sigma_t(a)\, b) = \varphi(b\, \sigma_{t+i\beta}(a)). \tag{10}$$

Consider now a Cuntz–Krieger C*-algebra $\mathcal{O}_A$. Any element in the $*$-algebra generated algebraically by the S_i can be written as a linear combination of monomials of the form $S_\mu S_\nu^*$, for multi-indices $\mu = (i_1, \ldots i_{|\mu|})$ and $\nu = (j_1, \ldots, j_{|\nu|})$. The subalgebra $\mathcal{F}_A$ is the AF-algebra generated by the elements of the form $S_\mu S_\nu^*$, for $|\mu| = |\nu|$. It is filtered by finite dimensional subalgebras $\mathcal{F}_k$, for $k \geq 0$, generated by the matrix units $E^i_{\mu,\nu} = S_\mu P_i S_\nu^*$, with $|\mu| = |\nu| = k$, where $P_i = S_i S_i^*$ are the range projections of the isometries S_i. The commutative algebra of functions on the Cantor set Λ_A of the subshift of finite type (Λ_A, T) associated to the Cuntz–Krieger algebra, is a maximal abelian subalgebra of $\mathcal{F}_A$, identified with the elements of the form $S_\mu S_\mu^*$ (*cf.* [5]).

In our case, one can consider a time evolution on $\mathcal{O}_{A_N}$ of the type considered in [7], with $\sigma_t^{u,h}(S_k) = \exp(it(u-h))\, S_k$, for $u = \log \eta_\beta = P(\beta)$, the topological pressure, *i.e.* the top eigenvalue of (8), and $h(x) = -\beta/2 \log |T'(x)|$. Thus, for all $t \in \mathbb{R}$, the function $\exp(-it\, h)$ acts on the elements of $\mathcal{O}_{A_N}$ by multiplication by an element in $C(E_N \times \mathbb{P})$.

Then a KMS$_1$ state φ for this time evolution satisfies the relation (*cf.* [7] Lemma 7.3)

$$\sum_k \varphi(S_k^*\, e^h a\, S_k) = e^u\, \varphi(a),$$

for $a \in \mathcal{O}_{A_N}$. For all $a = f \in C(E_N \times \mathbb{P})$, we have $\sum_k S_k^*\, e^h f\, S_k = \mathcal{L}_h(f)$, where the Ruelle transfer operator

$$\mathcal{L}_h(f)(x,s) = \sum_{(y,r)\in T^{-1}(x,s)} \exp(h(y))\, f(y,r)$$

is in fact the Perron–Frobenius operator (8). In particular, the KMS condition implies that the state φ restricts to $C(E_N \times \mathbb{P})$ to a probability measure μ satisfying $\mathcal{L}_h^*\mu = e^u\mu$. For $u = \log \eta_\beta$, the existence and uniqueness of such measure can be derived from the properties of the operator (8), along the lines of [12] [9].

Theorem 5.2 *For $\beta < 2\log r(A_N)/\log(N+1)$, there exists a unique KMS$_1$ state for the time evolution $\sigma_t^{P(\beta),-\beta/2\,\log|T'|}$ on the algebra $\mathcal{O}_{A_N}$. This restricts to the subalgebra $C(E_N \times \mathbb{P})$ to $f \mapsto \int f\, d\mu$ with the probability measure satisfying $\mathcal{L}_\beta^*\mu = \eta_\beta\mu$, for $\mathcal{L}_\beta$ the Perron–Frobenius operator* (8)*.*

Proof. Since by Lemma 5.1 the matrix A_N is irreducible and aperiodic, by Proposition 7.6 of [7], there is a surjective map of the set of KMS states to the set of probability measures satisfying $\mathcal{L}_h^*\mu = e^u\mu$. Uniqueness follows from Lemma 7.5 and Theorem 7.8 of [7], by showing that the estimate $var_0(h) < r(A_N)$ holds, where $r(A_N)$ is the spectral radius of the matrix A_N and $var_0(h) = \max h - \min h$ on $E_N \times \mathbb{P}$. This provides the range of values of β specified above, since on $E_N \times \mathbb{P}$ we have $var_0(-\beta/2 \log|T'|) = \log(N+1)^{\beta/2}$. The KMS state φ is obtained as in [7], by defining inductively compatible states

$$\varphi_k(a) = e^{-u} \sum_j \varphi_{k-1}(S_j^* e^{h/2} a e^{h/2} S_j) \quad \text{for} \quad a \in \mathcal{F}_k.$$

References

1. K. I. Babenko, *On a problem of Gauss*. Dokl. Akad. Nauk SSSR, Tom 238 (1978) No. 5, 1021–1024.
2. J. D. Barrow. *Chaotic behaviour and the Einstein equations.* In: Classical General Relativity, eds. W. Bonnor et al., Cambridge Univ. Press, Cambridge, 1984, 25–41.
3. V. Belinskii, I. M. Khalatnikov, E. M. Lifshitz. *Oscillatory approach to singular point in Relativistic cosmology.* Adv. Phys. 19 (1970), 525-551.
4. A. Connes, *Noncommutative Geometry*, Academic Press, 1994.
5. J. Cuntz, W. Krieger, *A class of C^*–algebras and topological Markov chains*, Invent. Math. 56 (1980) 251–268.
6. D. Hensley, *Continued fraction Cantor sets, Hausdorff dimension, and functional analysis*, J. Number Theory 40 (1992) 336–358.
7. D. Kerr, C. Pinzari, *Noncommutative pressure and the variational principle in Cuntz–Krieger–type C^*-algebras*, J.Funct.Anal. 188 (2002) 156–215.
8. I. M. Khalatnikov, E. M. Lifshitz, K. M. Khanin, L. N. Schur, Ya. G. Sinai. *On the stochasticity in relativistic cosmology.* J. Stat. Phys., 38:1/2 (1985), 97–114.
9. Yu.I. Manin, M. Marcolli, *Continued fractions, modular symbols, and noncommutative geometry*, Selecta Math. (New Ser.) Vol. 8 N.3 (2002) 475–520.

10. M. Marcolli, *Limiting modular symbols and the Lyapunov spectrum*, J.Number Theory Vol.98 N.2 (2003) 348–376.
11. D.H. Mayer, *Relaxation properties of the mixmaster universe*, Phys. Lett. A 121 (1987), no. 8-9, 390–394.
12. D.H. Mayer, *Continued fractions and related transformations.* In: Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces, Eds. T. Bedford et al., Oxford University Press, Oxford 1991, pp. 175–222.
13. C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, W H Freeman and Co. 1973.
14. M. Pollicott, H. Weiss, *Multifractal analysis of Lyapunov exponent for continued fraction and Manneville-Pomeau transformations and applications to Diophantine approximation*, Comm. Math. Phys. 207 (1999), no. 1, 145–171.